A bound on the scrambling index of a primitive matrix using Boolean rank

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Abstract

The scrambling index of an $n \times n$ primitive matrix A is the smallest positive integer k such that $A^k(A^t)^k = J$, where A^t denotes the transpose of A and J denotes the $n \times n$ all ones matrix. For an $m \times n$ Boolean matrix M, its *Boolean rank* $b(M)$ is the smallest positive integer b such that $M = AB$ for some $m \times b$ Boolean matrix A and $b \times n$ Boolean matrix B. In this paper, we give an upper bound on the scrambling index of an $n \times n$ primitive matrix M in terms of its Boolean rank $b(M)$. Furthermore we characterize all primitive matrices that achieve the upper bound.

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1 Introduction

For terminology and notation used here we follow [3]. A matrix A is called *nonnegative* if all its elements are nonnegative, and denoted by $A \geq 0$. A matrix A is called *positive* if all its elements are positive, and denoted by $A > 0$. For an $m \times n$ matrix A, we will denote its (i, j)-entry by A_{ij}, its ith

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row by $A_{i.}$, and its jth column by $A_{.j}$. For $m \times n$ matrices A and B, we say that B is dominated by A if $B_{ij} \leq A_{ij}$ for each i and j, and denote $B \leq A$. We denote the $m \times n$ all ones matrix by $J_{m,n}$ (and by J_n if $m = n$), The $m \times n$ all zeros matrix by $O_{m,n}$, the all ones n-vector by j_n, the $n \times n$ identity matrix by I_n, and its ith column by $e_i(n)$. The subscripts m and n will be omitted whenever their values are clear from the context.

For an $n \times n$ nonnegative matrix $A = (a_{ij})$, its digraph, denoted by $D(A)$, is the digraph with vertex set $V(D(A)) = \{1, 2, \cdots, n\}$, and (i, j) is an arc of $D(A)$ if and only if $a_{ij} \neq 0$. Then, for a positive integer $r \geq 1$, the (i, j)-th entry of the matrix A^r is positive if and only if $i \xrightarrow{r} j$ in the digraph $D(A)$. Since most of the time we are only interested in the existence of such walks, not the number of different directed walks from vertex i to vertex j, we interpret A as a Boolean $(0, 1)$-matrix, unless stated otherwise. A *Boolean* $(0,1)$*-matrix* is a matrix with only 0's and 1's as its entries. Using *Boolean arithmetic*, $(1 + 1 = 1,\ 0 + 0 = 0,\ 1 + 0 = 1)$, we have that AB and $A + B$ are Boolean $(0, 1)$-matrices if A and B are.

Let $D = (V, E)$ denote a *digraph* (directed graph) with vertex set $V = V(D)$, arc set $E = E(D)$ and order n. Loops are permitted but multiple arcs are not. A $u \rightarrow v$ *walk* in a digraph D is a sequence of vertices $u, u_1, \ldots, u_t, v \in V(D)$ and a sequence of arcs $(u, u_1), (u_1, u_2), \ldots, (u_t, v) \in E(D)$, where the vertices and arcs are not necessarily distinct. We shall use the notation $u \rightarrow v$ and $u \nrightarrow v$ to denote, respectively, that there is an arc from vertex u to vertex v and that there is no such an arc. Similarly, $u \xrightarrow{k} v$ and $u \overset{k}{\nrightarrow} v$ denote, respectively, that there is a directed walk of length k from vertex u to vertex v, and that there is no such a walk.

A digraph D is called *primitive* if for some positive integer t there is a walk of length exactly t from each vertex u to each vertex v. If D is primitive the smallest such t is called the *exponent* of D, denoted by $\exp(D)$. Equivalently, a square nonnegative matrix A of order n is called *primitive* if there exists a positive integer r such that $A^r > 0$. The minimum such r is called the *exponent* of A, and denoted by $\exp(A)$. Clearly $\exp(A) = \exp(D(A))$. There are numerous results on the exponent of primitive matrices [3].

The *scrambling index* of a primitive digraph D is the smallest positive integer k such that for every pair of vertices u and v, there exists some vertex $w = w(u, v)$ (dependent of u and v) such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in D. The scrambling index of D is denoted by $k(D)$. For $u, v \in V(D)$ $(u \neq v)$, we define the *local scrambling index of* u *and* v as

$$k_{u,v}(D) = \min\{k :\ u \xrightarrow{k} w \text{ and } v \xrightarrow{k} w \text{ for some } w \in V(D)\}.$$

Then

$$k(D) = \max_{u,v \in V(D)} \{k_{u,v}(D)\}.$$

An analogous definition for scrambling index can be given for nonnegative matrices. The *scrambling index* of a primitive matrix A, denoted by $k(A)$, is the smallest positive integer k such that any two rows of A^k have at least one positive element in a coincident position. The scrambling index of a primitive matrix A can also be equivalently defined as the smallest positive integer k such that $A^k(A^t)^k = J$, where A^t denotes the transpose of A. If A is the adjacency matrix of a primitive digraph D, then $k(D) = k(A)$. As a result, throughout the paper, where no confusion occurs, we use the digraph D and the adjacency matrix $A(D)$ interchangeably.

In [1] and [2], Akelbek and Kirkland obtained an upper bound on the scrambling index of a primitive digraph D in terms of the order and girth of D, and gave a characterization of the primitive digraphs with the largest scrambling index.

Theorem 1.1 **[1]** *Let D be a primitive digraph with n vertices and girth s. Then*

$$k(D) \leq n - s + \begin{cases} (\frac{s-1}{2})n, & \text{when } s \text{ is odd,} \\ (\frac{n-1}{2})s, & \text{when } s \text{ is even.} \end{cases}$$

When $s = n-1$, an upper bound on $k(D)$ in terms of the order of a primitive digraph D can be achieved [1]. We state the theorem in terms of primitive matrices below.

Theorem 1.2 **[1]** *Let A be a primitive matrix of order $n \geq 2$. Then*

$$k(A) \leq \left\lceil \frac{(n-1)^2+1}{2} \right\rceil. \tag{1}$$

Equality holds in (1) if and only if there is a permutation matrix P such that PAP^t is one of the following matrices

$$W_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad or \quad J_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad when\ n = 2,$$

$$W_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \textit{ when } n \geq 3.$$

The digraph $D(W_n)$ is called the Wielandt graph and denoted by $D_{n-1,n}$. It is a digraph with a Hamilton cycle $1 \to 2 \to \cdots \to n \to 1$ together with an arc from vertex $n-1$ to vertex 1. For simplicity, let $h_n = \left\lceil \frac{(n-1)^2+1}{2} \right\rceil$. The next proposition gives some information about the Wielandt graph $D_{n-1,n}$.

Proposition 1.3 [1] *For $D_{n-1,n}$, where $n \geq 3$,*

(a) $k_{n,\lfloor \frac{n}{2} \rfloor}(D_{n-1,n}) = h_n$, *and for all other pairs of vertices u and v of $D_{n-1,n}$, $k_{u,v}(D_{n-1,n}) < h_n$.*

(b) *There are directed walks from vertices n and $\lfloor \frac{n}{2} \rfloor$ to vertex 1 of length h_n, that is $n \xrightarrow{h_n} 1$ and $\lfloor \frac{n}{2} \rfloor \xrightarrow{h_n} 1$.*

For an $m \times n$ Boolean matrix M, we define its *Boolean rank* $b(M)$ to be the smallest positive integer b such that for some $m \times b$ Boolean matrix A and $b \times n$ Boolean matrix B, $M = AB$. The Boolean rank of the zero matrix is defined to be zero. $M = AB$ is called a *Boolean rank factorization* of M.

In [4], Gregory, Kirkland and Pullman obtained an upper bound on the exponent of primitive Boolean matrix in terms of Boolean rank.

Proposition 1.4 [4] *Suppose that $n \geq 2$ and that M is an $n \times n$ primitive Boolean matrix with $b(M) = b$. Then*

$$\exp(M) \leq (b-1)^2 + 2. \tag{2}$$

In [4], Gregory, Kirkland and Pullman also gave a characterization of the matrices for which equality holds in (2). In [5], Liu, You and Yu gave a characterization of primitive matrices M with Boolean rank b such that $\exp(M) = (b-1)^2 + 1$.

In this paper, we give an upper bound on the scrambling index of a primitive matrix M using Boolean rank $b = b(M)$, and characterize all Boolean primitive matrices that achieve the upper bound.

2 Main Results

We start with a basic result.

Lemma 2.1 *Suppose that A and B are $n \times m$ and $m \times n$ Boolean matrices respectively, and that neither has a zero line. Then*

(a) AB is primitive if and only if BA is primitive.

(b) If AB and BA are primitive, then

$$|k(AB) - k(BA)| \leq 1. \tag{3}$$

Proof. Part (a) was proved by Shao [6]. We only need to show part (b). Since AB and BA are primitive matrices, A and B has no zero rows. Then $AA^t \geq I_n$ and $BJ_nB^t = J_m$. Suppose $k(AB) = k$. By the definition of scrambling index

$$(AB)^k((AB)^t)^k = J_n.$$

Then

$$\begin{aligned}(BA)^{k+1}((BA)^t)^{k+1} &= B(AB)^k AA^t((AB)^t)^k B^t \geq B(AB)^k I_n((AB)^t)^k B^t \\ &= B(AB)^k((AB)^t)^k B^t = BJ_nB^t = J_m.\end{aligned}$$

Thus $k(BA) \leq k+1 = k(AB)+1$. The result follows by exchanging the roles of A and B. □

Proposition 2.2 [5] *Let M be an $n \times n$ primitive Boolean matrix, and $M = AB$ be a Boolean rank factorization of M. Then neither A nor B has a zero line.*

Theorem 2.3 *Let M be an $n \times n$ $(n \geq 2)$ primitive matrix with Boolean rank $b(M) = b$. Then*

$$k(M) \leq \left\lceil \frac{(b-1)^2+1}{2} \right\rceil + 1. \tag{4}$$

Proof. Let $M = AB$ be a Boolean rank factorization of M, where A and B are $n \times b$ and $b \times n$ Boolean matrices respectively. Then by Lemma 2.2 neither A nor B has a zero line. By lemma 2.1, we have

$$k(M) = k(AB) \leq k(BA) + 1.$$

Since BA is primitive and BA is a $b \times b$ matrix, by Theorem 1.2,

$$k(BA) \leq \left\lceil \frac{(b-1)^2+1}{2} \right\rceil,$$

from which Theorem 2.3 follows. □

From (1) we see that no matrix of full Boolean rank n can attain the upper bound in (4). Further, since the only $n \times n$ primitive Boolean matrix with Boolean rank 1 is J_n, no matrix of Boolean rank 1 can attain the upper bound in (4). Thus we may assume that $2 \leq b \leq n-1$.

For simplicity, let

$$h = \left\lceil \frac{(b-1)^2+1}{2} \right\rceil.$$

Recall from Theorem 1.2 that $k(W_b) = h$. We first make some observations about W_b. Recall that $D = D(W_b)$ is the Wielandt graph $D_{b-1,b}$ with b vertices.

Lemma 2.4 *If $b \geq 3$, then the zero entries of $(W_b)^{h-1}(W_b^t)^{h-1}$ occur only in the $(b, \lfloor\frac{b}{2}\rfloor)$ and $(\lfloor\frac{b}{2}\rfloor, b)$ positions.*

Proof. By Proposition 1.3 we know that $k_{b,\lfloor\frac{b}{2}\rfloor}(D_{b-1,b}) = h$, and for all other pairs of vertices u and v, $k_{u,v}(D_{b-1,b}) < h$. Therefore in W_b^{h-1} every pair of rows intersect with each other except rows b and $\lfloor\frac{b}{2}\rfloor$. Thus the only zero entries of $(W_b)^{h-1}(W_b^t)^{h-1}$ are in the $(b, \lfloor\frac{b}{2}\rfloor)$ and $(\lfloor\frac{b}{2}\rfloor, b)$ positions. □

For an $n \times n$ $(n \geq 2)$ matrix A, let $A(\{i_1, i_2\}, \{j_1, j_2\})$ be the submatrix of A that lies in the rows i_1 and i_2 and the columns j_1 and j_2.

Lemma 2.5 *For $b \geq 3$, $W_b^{h-1}(\{\lfloor\frac{b}{2}\rfloor, b\}, \{b-1, b\})$ is either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.*

Proof. By Proposition 1.3, we know that $k_{\lfloor\frac{b}{2}\rfloor,b}(D_{b-1,b}) = h$ and $\lfloor\frac{b}{2}\rfloor \xrightarrow{h} 1$ and $b \xrightarrow{h} 1$. From the digraph $D_{b-1,b}$, we know that the directed walks of length h from vertices $\lfloor\frac{b}{2}\rfloor$ and b to vertex 1 is either

$$\lfloor\frac{b}{2}\rfloor \xrightarrow{h-1} b-1 \xrightarrow{1} 1,$$

$$b \xrightarrow{h-1} b \xrightarrow{1} 1,$$

or

$$\lfloor\frac{b}{2}\rfloor \xrightarrow{h-1} b \xrightarrow{1} 1,$$

$$b \xrightarrow{h-1} b-1 \xrightarrow{1} 1.$$

For the first case, if $\lfloor \frac{b}{2} \rfloor \overset{h-1}{\rightarrow} b-1$ and $b \overset{h-1}{\rightarrow} b$, then $b \overset{h-1}{\not\rightarrow} b-1$ and $\lfloor \frac{b}{2} \rfloor \overset{h-1}{\not\rightarrow} b$. Otherwise it contradicts to $k_{\lfloor \frac{b}{2} \rfloor, b}(D_{b-1,b}) = h$. Similarly, for the second case if $\lfloor \frac{b}{2} \rfloor \overset{h-1}{\rightarrow} b$ and $b \overset{h-1}{\rightarrow} b-1$, then $b \overset{h-1}{\not\rightarrow} b$ and $\lfloor \frac{b}{2} \rfloor \overset{h-1}{\not\rightarrow} b-1$. The result follows by applying these to the matrix W_b^{h-1}. $\square$

Theorem 2.6 *Suppose M is an $n \times n$ primitive Boolean matrix with $3 \leq b = b(M) \leq n-1$. Then*

$$k(M) = \left\lceil \frac{(b-1)^2+1}{2} \right\rceil + 1$$

if and only if M has a boolean rank factorization $M = AB$, where A and B have the following properties:

(i) $BA = W_b$,
(ii) *some row of A is $e^t_{\lfloor \frac{b}{2} \rfloor}(b)$, some row of A is $e^t_b(b)$, and*
(iii) *no column of B is $e_{b-1}(b) + e_b(b)$.*

Proof. First suppose M is primitive with $k(M) = h+1$, and $M = \tilde{A}\tilde{B}$ is a Boolean rank factorization of M. By Lemma 2.1, $\tilde{B}\tilde{A}$ is primitive and $k(\tilde{B}\tilde{A}) \geq h$. But $\tilde{B}\tilde{A}$ is a $b \times b$ matrix. By Theorem 1.2, $k(\tilde{B}\tilde{A}) \leq h$. Therefore $k(\tilde{B}\tilde{A}) = h$. Also by Theorem 1.2, there is a permutation matrix P such that $P\tilde{B}\tilde{A}P^t = W_b$. Let $B = P\tilde{B}$ and $A = \tilde{A}P^t$. Then $AB = \tilde{A}P^tP\tilde{B} = \tilde{A}\tilde{B} = M$. Thus A and B satisfy condition (i).

Since M is primitive, we have $\sum_{i=1}^b A_{.i} = j_n = \sum_{i=1}^b B^t_{i.}$. Since $k(M) = h+1$, the matrix M^h must have two rows that do not intersect. Without lost of generality, suppose rows p and q of M^h do not intersect. Then entries in the (p,q) and (q,p) positions of $M^h(M^t)^h$ are zero. Since matrix B has no zero row, we have $BB^t \geq I_b$. Thus

$$\begin{aligned}
&M^h(M^t)^h \\
&= (AB)^h((AB)^t)^h = A(BA)^{h-1}BB^t((BA)^t)^{h-1}A^t \\
&= A(W_b)^{h-1}BB^t(W_b^t)^{h-1}A^t \\
&\geq A(W_b)^{h-1}I_b(W_b^t)^{h-1}A^t = A(W_b)^{h-1}(W_b^t)^{h-1}A^t \\
&= AZA^t \\
&= \left[J_{n,\lfloor \frac{b}{2} \rfloor - 1} \middle| \sum_{i=1}^{b-1} A_{.i} \middle| J_{n,b-\lfloor \frac{b}{2} \rfloor -1} \middle| \sum_{\substack{i=1 \\ i \neq \lfloor \frac{b}{2} \rfloor}}^{b} A_{.i} \right] A^t \\
&= j_n \left(\sum_{i=1}^{\lfloor \frac{b}{2} \rfloor - 1} A_{.i} \right)^t + \left(\sum_{i=1}^{b-1} A_{.i} \right) (A_{.\lfloor \frac{b}{2} \rfloor})^t + j_n \left(\sum_{i=\lfloor \frac{b}{2} \rfloor + 1}^{b-1} A_{.i} \right)^t + \left(\sum_{\substack{i=1 \\ i \neq \lfloor \frac{b}{2} \rfloor}}^{b} A_{.i} \right) (A_{.b})^t,
\end{aligned}$$

where $Z = (W_b)^{h-1}(W_b^t)^{h-1}$ is the $b \times b$ matrix which has zero entries only in the $(\lfloor \frac{b}{2} \rfloor, b)$ and $(b, \lfloor \frac{b}{2} \rfloor)$ positions. Since AZA^t is dominated by $M^h(M^t)^h$ and $M^h(M^t)^h$ has zero entries in the (p, q) and (q, p) positions, the entries in the (p, q) and (q, p) positions of AZA^t are also zero. Thus

$$\sum_{i=1}^{\lfloor \frac{b}{2} \rfloor - 1} A_{qi} + \left(\sum_{i=1}^{b-1} A_{pi} \right) A_{q\lfloor \frac{b}{2} \rfloor} + \sum_{i=\lfloor \frac{b}{2} \rfloor + 1}^{b-1} A_{qi} + \left(\sum_{\substack{i=1 \\ i \neq \lfloor \frac{b}{2} \rfloor}}^{b} A_{pi} \right) A_{qb} = 0 \tag{5}$$

and

$$\sum_{i=1}^{\lfloor \frac{b}{2} \rfloor - 1} A_{pi} + \left(\sum_{i=1}^{b-1} A_{qi} \right) A_{p\lfloor \frac{b}{2} \rfloor} + \sum_{i=\lfloor \frac{b}{2} \rfloor + 1}^{b-1} A_{pi} + \left(\sum_{\substack{i=1 \\ i \neq \lfloor \frac{b}{2} \rfloor}}^{b} A_{qi} \right) A_{pb} = 0. \tag{6}$$

Then $A_{qi} = 0$ and $A_{pi} = 0$ for $i = 1, \cdots, b-1$ and $i \neq \lfloor \frac{b}{2} \rfloor$. Substitute these back to (5) and (6), we have

$$A_{q\lfloor \frac{b}{2} \rfloor} A_{p\lfloor \frac{b}{2} \rfloor} + A_{qb} A_{pb} = 0. \tag{7}$$

If $A_{q\lfloor \frac{b}{2} \rfloor} \neq 0$, then $A_{p\lfloor \frac{b}{2} \rfloor} = 0$. Since every row of A is nonzero, we have $A_{pb} \neq 0$. By (7), $A_{qp} = 0$. Therefore some rows of A is $e^t_{\lfloor \frac{b}{2} \rfloor}(b)$ and some row of A is $e^t_b(b)$. This concludes (ii).

We claim B can not have a column which is equal to u. Otherwise, suppose some column of B is u. Since B has no zero row, by Proposition 2.2, $BB^t \geq I_b + uu^t$. Thus

$$\begin{aligned} M^h(M^t)^h &= (AB)^h((AB)^t)^h = A(BA)^{h-1}BB^t((BA)^t)^{h-1}A^t \\ &= A(W_b)^{h-1}BB^t(W_b^t)^{h-1}A^t \\ &\geq A(W_b)^{h-1}(I_b + uu^t)(W_b^t)^{h-1}A^t \\ &= A[(W_b)^{h-1}(W_b^t)^{h-1} + (W_b^{h-1}u)(W_b^{h-1}u)^t]A^t. \end{aligned}$$

By lemma 2.4, $W_b^{h-1}(\{\lfloor \frac{b}{2} \rfloor, b\}, \{b-1, b\})$ is either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $W_b^{h-1}u \geq e_{\lfloor \frac{b}{2} \rfloor}(b) + e_b(b)$. By Lemma 2.4, the zero entries of $W_b^{h-1}(W_b^t)^{h-1}$ are in the $(b, \lfloor \frac{b}{2} \rfloor)$ and $(\lfloor \frac{b}{2} \rfloor, b)$ positions. Therefore $W_b^{h-1}(W_b^t)^{h-1} + (W_b^{h-1}u)(W_b^{h-1}u)^t = J_b$. Since A has no zero lines, we have $M^h(M^t)^h = AJ_bA^t = J_n$, which is a contradiction to $k(M) = h+1$. This proves (iii).

Finally, suppose that $M = AB$ is a Boolean rank factorization of M and A and B satisfy (i), (ii) and (iii). By Lemma 2.1(a) and Theorem 1.2, the matrix M is primitive and $k(M) \leq h+1$ by Lemma 2.1(b) and . But it follows from Lemma 2.4 and conditions (i), (ii) and (iii) that M^h has zero entries. So we conclude that $k(D) = h+1$. $\square$

Next we will reinterpret conditions (i), (ii) and (iii) of Theorem 2.6 to show that if $k(M) = h+1$, then M is one of the three basic types of matrices in Theorem 2.7.

Table 1 $(b \geq 3)$

$$M_1 = \left[\begin{array}{cccccc|c} 0 & J & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J & 0 & J \\ 0 & 0 & \cdots & 0 & 0 & J & 0 \\ J & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline J & 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right] \qquad M_2 = \left[\begin{array}{cccccc|c} 0 & J & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J & 0 & J \\ 0 & 0 & \cdots & 0 & 0 & J & 0 \\ J & 0 & \cdots & 0 & 0 & 0 & 0 \\ \hline J & 0 & \cdots & 0 & 0 & J & 0 \end{array}\right]$$

$$M_3 = \left[\begin{array}{cccccc|cc} 0 & J & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J & 0 & J & J \\ 0 & 0 & \cdots & 0 & 0 & J & 0 & 0 \\ J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \hline J & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ J & 0 & \cdots & 0 & 0 & J & 0 & 0 \end{array}\right]$$

Theorem 2.7 *Suppose M is an $n \times n$ Boolean matrix with $b(M) = b$, where $3 \leq b \leq n-1$. Then M is primitive with $k(M) = h+1$ if and only if there is a permutation matrix P such that PMP^t has one of the forms in Table* 1.

In Table 1 the rows and columns of M_1, M_2 and M_3 are partitioned conformally, so that each diagonal block is square, and the top left hand submatrix common to each has b blocks in its partitioning.

Proof. Suppose M is primitive, $b \geq 3$, and $k(M) = h+1$. Then by Theorem 2.6(i), M has a Boolean rank factorization $M = AB$ such that $BA = W_b$. Since A has no zero row, each column of B is dominated by a column of W_b. Similarly, each row of A is dominated by a row of W_b. Thus each column of B is in the set $S_1 = \{e_1(b), e_2(b), \cdots, e_b(b), u\}$, where $u = e_{b-1}(b) + e_b(b)$. Similarly, each row of A is in the set $S_2 = \{e_1^t(b), e_2^t(b), \cdots, e_b^t(b), v^t\}$, where $v = e_1(b) + e_b(b)$. But by Theorem 2.6(iii), no column of B is u. Hence each column of B is in the set of $S_1' = \{e_1(b), e_2(b), \cdots, e_b(b)\}$.

Next, we note that for each $1 \leq i \leq b$, the product $B_{.i}A_{i.}$ is dominated by W_b. Since each $B_{.i}$ and $A_{i.}$ must be in S_1' and S_2 respectively and $(B_{.i}, A_{i.})$ must be one of the following pairs: (e_i, e_{i+1}^t), $1 \leq i \leq b-1$, (e_{b-1}, e_1^t), (e_b, e_1), or (e_{b-1}, v^t), where $e_i = e_i(b)$ for any $i \in \{1, 2, \cdots, b\}$. Thus, for each i, $1 \leq i \leq b-1$, $(e_i, e_{i+1}^t) = (B_{.k_i}, A_{k_i.})$ for some k_i. Some outer product $B_{.j}A_{j.}$ has a 1 in the $(b,1)$ position, hence $(B_{.k_b}, A_{k_b.}) = (e_b, e_1^t)$ for some k_b. Finally some outer product $B_{.j}A_{j.}$ must have a 1 in the $(b-1, 1)$ position, hence for some k_{b+1}, $(B_{.k_{b+1}}, A_{k_{k+1}.})$ is one of (e_{b-1}, e_1^t) or (e_{b-1}, v^t). It follows from the above argument that there is an $n \times n$ permutation matrix Q such that

$$BQ^t = \left[\bar{B} | \tilde{B}\right] \quad \text{and} \quad QA = \left[\begin{array}{c} \bar{A} \\ \hline \tilde{A} \end{array}\right],$$

where

$$\bar{B} = \left[e_1 j_{n_1}^t | e_2 j_{n_2}^t | \cdots | e_b j_{n_b}^t\right] \quad \text{and} \quad \bar{A} = \left[\begin{array}{c} j_{n_1} e_2^t \\ \hline j_{n_2} e_3^t \\ \hline \cdots \\ \hline j_{n_{b-1}} e_b^t \\ \hline j_{n_b} e_1^t \end{array}\right]$$

for some $n_1, \cdots, n_b \geq 1$, and where each $(\tilde{B}_{.i}, \tilde{A}_{i.})$ is one of (e_{b-1}, e_1^t) or (e_{b-1}, v^t). Thus $\tilde{B}$ and $\tilde{A}$ can be one of the following pairs of matrices:

$$\tilde{B}_1 = e_{b-1} j_{m_1}^t, \ \tilde{A}_1 = j_{m_1} e_1^t \ \text{ for some } m_1 \geq 1;$$
$$\tilde{B}_2 = e_{b-1} j_{m_2}^t, \ \tilde{A}_2 = j_{m_2} v^t \ \text{ for some } m_2 \geq 1;$$
$$\tilde{B}_3 = \left[e_{b-1} j_{m_3}^t | e_{b-1} j_{p_3}^t\right], \ \tilde{A}_3 = \left[\begin{array}{c} j_{m_3} e_1^t \\ \hline j_{p_3} v^t \end{array}\right] \ \text{ for some } m_3, p_3 \geq 1.$$

It is now readily verified that

$$\left[\frac{\bar{A}}{\tilde{A}_i}\right]\left[\bar{B}|\tilde{B}_i\right] = M_i \quad \text{for } 1 \le i \le 3,$$

so that QMQ^t is one of the matrices in Table 1.

Finally, since the Boolean rank factorization

$$M_i = \left[\frac{\bar{A}}{\tilde{A}_i}\right]\left[\bar{B}|\tilde{B}_i\right]$$

satisfies conditions (i), (ii) and (iii) of Theorem 2.6, each M_i is primitive and $k(M) = h + 1$. □

When $b(M) = 2$, we have the following result.

Theorem 2.8 *Suppose M is an $n \times n$ primitive Boolean matrix with $b(M) = b = 2$. Then $k(M) = 2$ if and only if M has a boolean rank factorization $M = AB$, where A and B have the following properties:*

(i) *$BA = W_2$ or $BA = J_2$,*

(ii) *some row of A is $e_1^t(2)$, some row of A is $e_2^t(2)$, and*

(iii) *no column of B is $e_1(2) + e_2(2)$.*

Proof. First suppose M is primitive with $k(M) = 2$, and $M = \tilde{A}\tilde{B}$ is a Boolean rank factorization of M. By Lemma 2.1, $\tilde{B}\tilde{A}$ is primitive and $k(\tilde{B}\tilde{A}) \ge 1$. But $\tilde{B}\tilde{A}$ is a 2×2 matrix. By Theorem 1.2, $k(\tilde{B}\tilde{A}) \le 1$. Therefore $k(\tilde{B}\tilde{A}) = 1$. Also by Theorem 1.2, there is a permutation matrix P such that $P\tilde{B}\tilde{A}P^t = W_2$ or $P\tilde{B}\tilde{A}P^t = J_2$. Let $B = P\tilde{B}$ and $A = \tilde{A}P^t$. Then $AB = \tilde{A}P^tP\tilde{B} = \tilde{A}\tilde{B} = M$. Thus A and B satisfy condition (i).

Proof of the conditions (ii) and (iii) are similar to the proof of Theorem 2.6. □

By a similar argument, we can reinterpret conditions (i), (ii) and (iii) of Theorem 2.8 to show that if M satisfies $k(M) = 2$, then M is one of the 21 basic types of matrices which we will show in the following.

Theorem 2.9 *Suppose M is an $n \times n$ Boolean matrix with $b(M) = b = 2$. Let $M = AB$ be a Boolean rank factorization. Then M is primitive with $k(M) = 2$ if and only if there is a permutation matrix P such that PMP^t has one of the*

forms in Table 2 if $BA = W_2$ or PMP^t has one of the forms in Table 3 if $BA = J_2$.

In Table 2 and Table 3 the rows and columns of each matrix are partitioned conformally, so that each diagonal block is square.

Table 2 $(b = 2)$

$$\left[\begin{array}{cc|c} 0 & J & 0 \\ J & 0 & J \\ \hline J & 0 & J \end{array}\right], \quad \left[\begin{array}{cc|c} 0 & J & 0 \\ J & 0 & J \\ \hline J & J & J \end{array}\right], \quad \left[\begin{array}{cc|cc} 0 & J & 0 & 0 \\ J & 0 & J & J \\ \hline J & 0 & J & J \\ J & J & J & J \end{array}\right].$$

Table 3 $(b = 2)$

$$\left[\begin{array}{cccc} J & J & 0 & 0 \\ 0 & 0 & J & J \\ J & J & 0 & 0 \\ 0 & 0 & J & J \end{array}\right], \quad \left[\begin{array}{cccc|c} J & J & 0 & 0 & J \\ 0 & 0 & J & J & 0 \\ J & J & 0 & 0 & J \\ 0 & 0 & J & J & 0 \\ \hline J & J & J & J & J \end{array}\right], \quad \left[\begin{array}{cccc|c} J & J & 0 & 0 & 0 \\ 0 & 0 & J & J & J \\ J & J & 0 & 0 & 0 \\ 0 & 0 & J & J & J \\ \hline J & J & J & J & J \end{array}\right], \quad \left[\begin{array}{cccc|cc} J & J & 0 & 0 & J & 0 \\ 0 & 0 & J & J & 0 & J \\ J & J & 0 & 0 & J & 0 \\ 0 & 0 & J & J & 0 & J \\ \hline J & J & J & J & J & J \\ J & J & J & J & J & J \end{array}\right],$$

$$\left[\begin{array}{ccc} J & J & 0 \\ 0 & 0 & J \\ J & J & J \end{array}\right], \quad \left[\begin{array}{ccc|c} J & J & 0 & J \\ 0 & 0 & J & 0 \\ J & J & J & J \\ \hline J & J & J & J \end{array}\right], \quad \left[\begin{array}{ccc|c} J & J & 0 & 0 \\ 0 & 0 & J & J \\ J & J & J & J \\ \hline J & J & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} J & J & 0 & 0 \\ 0 & 0 & J & J \\ J & J & J & J \\ \hline 0 & 0 & J & J \end{array}\right],$$

$$\left[\begin{array}{ccc|cc} J & J & 0 & J & 0 \\ 0 & 0 & J & 0 & J \\ J & J & J & J & J \\ \hline J & J & J & J & J \\ J & J & 0 & J & 0 \end{array}\right], \quad \left[\begin{array}{ccc|cc} J & J & 0 & J & 0 \\ 0 & 0 & J & 0 & J \\ J & J & J & J & J \\ \hline J & J & J & J & J \\ 0 & 0 & J & 0 & J \end{array}\right],$$

$$\begin{bmatrix} J & J & J \\ J & 0 & 0 \\ 0 & J & J \end{bmatrix}, \quad \left[\begin{array}{ccc|c} J & J & J & J \\ J & 0 & 0 & J \\ 0 & J & J & 0 \\ \hline J & 0 & 0 & J \end{array}\right], \quad \left[\begin{array}{ccc|c} J & J & J & J \\ J & 0 & 0 & J \\ 0 & J & J & 0 \\ \hline 0 & J & J & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} J & J & J & J \\ J & 0 & 0 & 0 \\ 0 & J & J & J \\ \hline J & J & J & J \end{array}\right],$$

$$\left[\begin{array}{ccc|cc} J & J & J & J & J \\ J & 0 & 0 & J & 0 \\ 0 & J & J & 0 & J \\ \hline J & 0 & 0 & J & 0 \\ J & J & J & J & J \end{array}\right], \quad \left[\begin{array}{ccc|cc} J & J & J & J & J \\ J & 0 & 0 & J & 0 \\ 0 & J & J & 0 & J \\ \hline 0 & J & J & 0 & J \\ J & J & J & J & J \end{array}\right], \quad \begin{bmatrix} J & J & J & J \\ J & J & J & J \\ J & 0 & J & 0 \\ 0 & J & 0 & J \end{bmatrix}, \quad \begin{bmatrix} J & J & J & J \\ J & J & J & J \\ 0 & J & 0 & J \\ J & 0 & J & 0 \end{bmatrix}.$$

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